New Bound Estimation of Nonnegative Matrix Spectral Radius

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Abstract: In this paper we give an estimate of the upper and lower bounds on the spectral radius of a nonnegative matrix and proved this theorems,it has improved accurary of the spectral radius estimate. In addition,the validity and precision of these estimation are tested.

Keywords: Nonnegative matrix; Spectral radius; Estimation

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1. Introduction

Nonnegative matrix theory is widely applied in numerous fields such as numerical analysis, graph theory, linear programming, management science, and automation control. A matrix $A = (a_{ij})_{n\times n}$ is termed a nonnegative matrix if $a_{ij} \geq 0$. If the eigenvalues of $A \in C^{n\times n}$ are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then $\rho(A) = \max\limits_{1\leq i\leq n} |\lambda_i|$ is referred to the spectral radius of A. Currently, there is a well-known conclusion for $\rho(A)$ estimating the bound of using the elements of the matrix A :

Conclusion 1(Frobenius): Let $A = (a_{ij})_{n\times n} \in R^{n\times n}$ is a nonnegative matrix, and let $\rho(A)$ is spectral radius of A. Let r_i and c_i denote respectively the i row sums and i column sums of A.Then, $\min\limits_{1\leq i\leq n} r_i \leq \rho(A) \leq \max\limits_{1\leq i\leq n} r_i$, $\min\limits_{1\leq i\leq n} c_i \leq \rho(A) \leq \max\limits_{1\leq i\leq n} c_i$.

Conclusion 2(Wielandt): Let $A = (a_{ij})_{n\times n}$ is n order nonnegative matrix, X is n-dimensional column vector, and X_i is the i component. Then,

$$\min_{1\leq i\leq n}\left\{\frac{(AX)_i}{X_i}\right\} \leq \rho(A) \leq \max_{1\leq i\leq n}\left\{\frac{(AX)_i}{X_i}\right\}$$

Lemma1: Let A is n order matrix, and let A^T is the transpose of A. Let λ is the eigenvalue of A, and let $(x_1, x_2 \cdots, x_n)^T$ and $(y_1, y_2 \cdots, y_n)^T$ is the eigenvectors of A and A^T corresponding to λ the eigenvalue . Then, $\lambda\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} x_i c_i(A)$, $\lambda\sum\limits_{i=1}^{n} y_i = \sum\limits_{i=1}^{n} x_i r_i(A)$, Let r_i and c_i denote the i row sums and i column sums of A.

Lemma2: Let $q_{1,} q_2 \cdots, q_n$ are positive number, and let $p_{1,} p_2 \cdots, p_n$ are arbitrary real numbers. Then

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$\min_i \left\{ \frac{p_i}{q_i} \right\} \le \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i} \le \max_i \left\{ \frac{p_i}{q_i} \right\}$. Equality holds if and only if all the $\frac{p_i}{q_i}$ ratios are equal.

Lemma3: Let $\boldsymbol{A}$ is a n order nonnegative matrix. If there are several rows (or columns) of $\boldsymbol{A}$, denoted by the indices $n_{1,} n_2 \cdots, n_s$, and we delete $\boldsymbol{A}$ rows (or columns) and the corresponding columns (or rows) to obtain the matrix $\overline{\boldsymbol{A}}$, then $\rho(\boldsymbol{A}) = \rho(\overline{\boldsymbol{A}})$.

The characteristic of Conclusion 1 is that it is computationally simple but not very accurate, while Conclusion 2 improves the accuracy somewhat, yet still falls short of ideal. This paper presents two new bounds for [or "the spectral radius" if is understood in context], enhancing the precision of spectral radius estimation. All matrices mentioned hereinafter are invertible.

2. Main Results

Theorem 1: Let $\rho(\boldsymbol{A})$ is the spectral radius of the nonnegative matrix $\boldsymbol{A}$, Let r_i and c_i denote respectively the i row sums and i column sums of $\boldsymbol{A}$, $\boldsymbol{B} = (\boldsymbol{A}+\boldsymbol{I})^{\mathrm{n}-1}$. Then, for any positive integer k , we have:

$$\min_{\mathrm{i}} \left\{ \frac{\mathrm{c_i}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{k}})}{\mathrm{c_i}(\boldsymbol{B}^{\mathrm{k}})} \right\} \le \rho(\boldsymbol{A}) \le \max_{\mathrm{i}} \left\{ \frac{\mathrm{c_i}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{k}})}{\mathrm{c_i}(\boldsymbol{B}^{\mathrm{k}})} \right\} \min_{\mathrm{i}} \left\{ \frac{\mathrm{r_i}(\hat{\boldsymbol{A}}\boldsymbol{B}^{\mathrm{k}})}{\mathrm{r_i}(\boldsymbol{B}^{\mathrm{k}})} \right\} \le \rho(\boldsymbol{A}) \le \max_{\mathrm{i}} \left\{ \frac{\mathrm{r_i}(\quad ^{\mathrm{k}})}{\mathrm{r_i}(\boldsymbol{B}^{\mathrm{k}})} \right\}.$$

Proof: Let $(x_1, x_2 \cdots, x_n)^T \ge 0$, $(y_1, y_2 \cdots, y_n)^T \ge 0$ is the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}}$ corresponding to the eigenvalue ρ , respectively $\sum_{i=1}^{n} x_i = 1$, $\sum_{i=1}^{n} y_i = 1$, Since

$$(\boldsymbol{B}^{\mathrm{k}})\boldsymbol{X} = (\boldsymbol{A}+\boldsymbol{I})^{\mathrm{k(n-1)}}\boldsymbol{X} = (\rho+1)^{\mathrm{k(n-1)}},$$

we have $\boldsymbol{A}(\boldsymbol{B}^k)\boldsymbol{X} = \left[\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I})^{k(n-1)}\right]\boldsymbol{X} = \rho(\rho+1)^{k(n-1)}\boldsymbol{X}$.

By Lemma 1 $\rho(\rho+1)^{k(n-1)} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i r_i(\boldsymbol{A}\boldsymbol{B}^k)$, and $(\rho+1)^{k(n-1)} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i r_i(\boldsymbol{B}^k)$. Dividing these two inequalities, we obtain $\rho = \frac{\sum_{i=1}^{n} x_i r_i(\boldsymbol{A}\boldsymbol{B}^k)}{\sum_{i=1}^{n} x_i r_i(\boldsymbol{B}^k)}$. By Lemma 2, it follows that

$$\min_i \left\{ \frac{x_i r_i(\boldsymbol{A}\boldsymbol{B}^k)}{x_i r_i(\boldsymbol{B}^k)} \right\} \le \rho(\boldsymbol{A}) \le \max_i \left\{ \frac{x_i r_i(\boldsymbol{A}\boldsymbol{B}^k)}{x_i r_i(\boldsymbol{B}^k)} \right\},$$

i.e. $\min_i \left\{ \frac{r_i(\boldsymbol{A}\boldsymbol{B}^k)}{r_i(\boldsymbol{B}^k)} \right\} \le \rho(\boldsymbol{A}) \le \max_i \left\{ \frac{r_i(\boldsymbol{A}\boldsymbol{B}^k)}{r_i(\boldsymbol{B}^k)} \right\}$. Similarly, it can be proven that $\min_i \left\{ \frac{c_i(\boldsymbol{A}\boldsymbol{B}^k)}{c_i(\boldsymbol{B}^k)} \right\} \le \rho(\boldsymbol{A}) \le \max_i \left\{ \frac{c_i(\boldsymbol{A}\boldsymbol{B}^k)}{c_i(\boldsymbol{B}^k)} \right\}$. Specifically, when $k=0$, Theorem 1 reduces to Conclusion 1.

Corollary: Let $\rho(A)$ is the spectral radius of the nonnegative matrix A, and let r_i is the i row sum of A (where is the identity matrix). Then $B=(A+I)^{n-1}$, $c=(A^3+A^2+A+I)^{n-1}$ for any positive integer k,

$$\max_i\left\{\frac{r_i(A^kC^k)_i}{r_i(A^{k-1}C^k)}\right\}\le\max_i\left\{\frac{r_i(AB^k)}{r_i(B^k)}\right\},\quad \min_i\left\{\frac{r_i(\tilde{A}^kC^k)_i}{r_i(A^{k-1}C^k)}\right\}\ge\min_i\left\{\frac{r_i(\quad ^k)}{r_i(B^k)}\right\}$$

Proof: Let $M=A^{k-1}=(m_{ij})$. Then, by Theorem 1 and its corollary, we have...

$$\max_i\frac{r_i(A^kB^k)}{r_i(A^{k-1}B^k)}=\max_i\frac{r_i(A^{k-1}AB^k)}{r_i(A^{k-1}B^k)}=\max_i\frac{\sum_i m_{ij}r_j(AB^k)}{\sum_i m_{ij}r_j(B^k)}\le\max_i\max_i\frac{r_i(AB^k)}{r_i(B^k)}=\max_i\frac{r_i(AB^k)}{r_i(B^k)}$$

Similarly, it can be proven $\min_i\left\{\frac{r_i(A^kC^k)_i}{r_i(A^{k-1}C^k)}\right\}\ge\min_i\left\{\frac{r_i(AB^k)}{r_i(B^k)}\right\}$.

due to $c=(A^3+A^2+A+I)^{n-1}=(A+I)^{n-1}(A^2+I)^{n-1}$, 令 $H=(A^2+I)^{n-1}=(h_{ij})$ so

$$\max_i\frac{r_i(A^kC^k)}{r_i(A^{k-1}C^k)}=\max_i\frac{r_i(A^k(A+I)^{k(n-1)}(A^2+I)^{k(n-1)})}{r_i(A^{k-1}(A+I)^{k(n-1)}(A^2+I)^{k(n-1)})}$$

$$=\max_i\frac{r_i(HA^kB^k)}{r_i(HA^{k-1}B^k)}=\max_i\frac{\sum_j h_{ij}r_j(A^kB^k)}{\sum_j h_{ij}r_j(A^{k-1}B^k)}\le\max_i\max_i\frac{r_i(A^kB^k)}{r_i(A^{k-1}B^k)}=\max_i\frac{r_i(A^kB^k)}{r_i(A^{k-1}B^k)}=\max_i\frac{r_i(AB^k)}{r_i(B^k)}$$

Similarly, it can be proven $\min_i\frac{r_i(A^kC^k)}{r_i(A^{k-1}C^k)}\ge\min_i\frac{r_i(AB^k)}{r_i(A^{k-1}B^k)}\ge\min_i\frac{r_i(AB^k)}{r_i(B^k)}$.

Combining Theorem 1 and its inference, the following conclusion can be drawn:

$$\min_i\frac{r_i(AB^k)}{r_i(B^k)}\le\min_i\frac{r_i(A^kC^k)}{r_i(A^{k-1}C^k)}\le\rho(A)\le\max_i\frac{r_i(A^kC^k)}{r_i(A^{k-1}C^k)}\le\max_i\frac{r_i(AB^k)}{r_i(B^k)}.$$

Lemma 4: Let A is a nonnegative matrix. Then $\gamma_x\le\rho(A)\le\delta_x$, for any given nonnegative vector $x=(x_1,x_2,\cdots,x_n)^T$, it holds that $\delta_x=\max_{x_i>0}\left\{\frac{\sum_j a_{ij}x_j}{x_i}\right\}$, $\gamma_x=\min_{x_i>0}\left\{\frac{\sum_j a_{ij}x_j}{x_i}\right\}$. when taking $x^*=(r_1(A),r_2(A),\cdots,r_n(A))^T$ and $y^*=(c_1(A),c_2(A),\cdots,c_n(A))^T$, we have $\max\left\{\gamma_{x^*},\gamma_{y^*}\right\}\le\min\left\{\delta_{x^*},\delta_{y^*}\right\}$.

Proof: Since $Ax=\rho(A)x$, it follows that $\sum_{j=1}^n a_{ij}x_j=\rho(A)x_i$, i.e $\rho(A)=\frac{1}{x_i}\sum_{j=1}^n a_{ij}x_j$. Given $x=(x_1,x_2,\cdots,x_n)^T>0$. Let $D=diag(x_1,x_2,\cdots,x_n)$, then D is an invertible matrix and $D^{-1}=diag(x_1^{-1},x_2^{-1},\cdots,x_n^{-1})$.

Therefore $\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}=\begin{pmatrix} a_{11} & a_{12}\frac{x_2}{x_1} & \cdots & a_{1n}\frac{x_n}{x_1} \\ a_{21}\frac{x_1}{x_2} & a_{22} & \cdots & a_{2n}\frac{x_n}{x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}\frac{x_1}{x_n} & a_{n2}\frac{x_n}{x_1} & \cdots & a_{nn} \end{pmatrix}$, because similar matrices have the same spectral radius. $\rho(\boldsymbol{A})=\rho(\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D})$ By Conclusion 1, $\gamma_x \le \rho(\boldsymbol{A}) \le \delta_x$. Taking and $\boldsymbol{x}^*=(r_1(\boldsymbol{A}),r_2(\boldsymbol{A}),\cdots,r_n(\boldsymbol{A}))^T$, $\boldsymbol{y}^*=(c_1(\boldsymbol{A}),c_2(\boldsymbol{A}),\cdots,c_n(\boldsymbol{A}))^T$, we have $\gamma_{x^*} \le \rho(\boldsymbol{A}) \le \delta_{x^*}$. Since $\rho(\boldsymbol{A})=\rho(\boldsymbol{A}^T)$, it follows that

$$\max\left\{\gamma_{x^*},\gamma_{y^*}\right\} \le \rho(\boldsymbol{A}) \le \min\left\{\delta_{x^*},\delta_{y^*}\right\}.$$

Combining this with Conclusion 1, we have the following theorem:

Theorem 2: Let $\boldsymbol{A}$ is a nonnegative matrix. Then, for any given nonnegative vector $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^T$, it holds that $\max\left\{\gamma,\gamma^T\right\} \le \max\left\{\gamma_{x^*},\gamma_{y^*}\right\} \le \rho(\boldsymbol{A}) \le \min\left\{\delta_{x^*},\delta_{y^*}\right\} \le \min\left\{R,R^T\right\}$.

Here, R and R^T denote the maximum row sum and column sum, respectively, while γ,γ^T denote the minimum row sum and column sum, respectively. Theorem 2 provides a more accurate estimate of $\rho(\boldsymbol{A})$ compared to Theorem 1, and it is also more operationally feasible, making it a convenient algorithm.

3. Numerical Example

Example: Consider the following 3×3 matrix for estimating the spectral radius of a nonnegative matrix $\boldsymbol{A}=\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 1 & 1 \end{pmatrix}$.

It is easy to calculate that $\gamma=4$, $\gamma^T=5$, $R=8$, $R^T=7$, $\boldsymbol{x}^*=(4,8,6)^T$, $\boldsymbol{y}^*=(7,5,6)^T$. By computing $\rho(\boldsymbol{A})=5.74165738\cdots$, the estimate based on Conclusion 1 using rows is $4\le\rho(\boldsymbol{A})\le 8$. The estimate based on Conclusion 2 (Wielandt) using rows is $5\le\rho(\boldsymbol{A})\le 6.25$. According to Theorem 1, the estimate using rows when is $k=3$, $5.5019\le\rho(\boldsymbol{A})\le 6.0538$. According to Theorem 2, we obtain $5.6\le\rho(\boldsymbol{A})\le 5.857143$. Through this example, it is evident that Theorem 2 provides a more accurate estimate of the spectral radius of a nonnegative matrix, and the range is closer to $\rho(\boldsymbol{A})$. It can be seen that the estimation of the spectral radius of a nonnegative matrix by Theorem 2 is simple, feasible, and desirable, which furthers our understanding of $\rho(\boldsymbol{A})$.

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